NO ARBITRAGE PRICING ON CALL OPTIONS

LARRY, ZHIRONG LI

"Call options on the same underlying asset with the same maturity, with strikes $K_1 < K_2 < K_3$, are trading for C_1, C_2 and C_3 (no bid-ask spread) with $C_1 > C_2 > C_3$. Find necessary and sufficient conditions on the prices C_1, C_2 and C_3 such that no arbitrage exists corresponding to a portfolio made of positions in the three options."

- Type-1 Arbitrage is a trading strategy that generates a strictly positive cash flow between 0 and T in at least one state with positive probability and does not require an outflow of funds at any date, that is a trading strategy that produces something from nothing. A simple example of this kind of arbitrage is the opportunity to borrow and lend at two different rates of interest.
- Type-2 Arbitrage generates a net future cash flow of at least zero for sure, with the arbitrageur getting his profits up front. This kind of arbitrage is referred to as free lunch. The simultaneous purchase and sale of the same or essentially similar security in two different markets for advantageously different prices may illustrate this case.

Method 1 (Brute force Approach): An arbitrage exists if and only if a no-cost portfolio can be set up with non-negative payoff at maturity regardless of the price of the underlying asset at maturity, and such that the probability of a strictly positive payoff is greater than zero.

Consider a portfolio made of positions in the three options with value 0 at inception, and let $x_i > 0$ be the size of the portfolio position in the option with strike $K_i, i = 1, 2, 3$. Let $S = S(T)$ be the value of the underlying asset at maturity. For no-arbitrage to occur, there are three possibilities. (x_1 has to be positive otherwise, it is always possible to have negative payoff when $S \in [K_1, K_2]$. if $x_i > 0, \forall i \in \{1, 2, 3\}$, the payoff can not be negative.)

Portfolio 1: Long the K_1-option, short the K_2-option, long the K_3-option.

Arbitrage exists if we can find $x_i > 0, \forall i \in \{1, 2, 3\}$ such that

$$x_1 C_1 - x_2 C_2 + x_3 C_3 = 0$$

and

$$f(S) = x_1 (S - K_1)^+ - x_2 (S - K_2)^+ + x_3 (S - K_3)^+ \geq 0, \forall S \geq 0$$

Notice that the above inequality is piecewise linear in S thus the minimum can take at the extreme points. When $S \leq K_1$, $f(S) = 0$. As S increases from K_1 to K_2, $f(S)$ increases as well. When $S \in [K_2, K_3]$, the second term comes into existence and $f(S)$ may decrease when $x_1 < x_2$. When $S \in [K_3, \infty]$, in order for $f(S) \geq 0$, the slope must be non-negative, i.e., $x_1 - x_2 + x_3 \geq 0$. In this situation, it is sufficient for $f(S) \geq 0$ when the possible minimum occurring at K_3 to be no less than zero. That is,

$$x_1 (K_3 - K_1) - x_2 (K_3 - K_2) \geq 0$$

Therefore,

$$x_3 = x_2 \frac{C_2}{C_3} - x_1 \frac{C_1}{C_3} > 0$$

Date: June 17, 2012.

and

$$\frac{C_1 - C_3}{C_1 - C_2} \leq \frac{x_2}{x_1} \leq \frac{K_3 - K_1}{K_3 - K_2}$$

Portfolio 2: Long the K_1-option, short the K_2-option, short the K_3-option.

Arbitrage exists if we can find $x_i > 0, \forall i \in \{1, 2, 3\}$ such that

$$x_1 C_1 - x_2 C_2 + x_3 C_3 = 0$$

and

$$f(S) = x_1 (S - K_1)^+ - x_2 (S - K_2)^+ - x_3 (S - K_3)^+ \geq 0, \forall S \geq 0$$

Similarly, we have

$$x_1 - x_2 - x_3 \geq 0$$

and

$$x_1 (K_3 - K_1) - x_2 (K_3 - K_2) \geq 0$$

Notice that

$$x_2 + x_3 \leq x_1 = x_2 \frac{C_2}{C_1} - x_3 \frac{C_3}{C_1} < x_2 + x_3$$

which means arbitrage can not exist when long K_1-option and short the other two.

Portfolio 3: Long the K_1-option, long the K_2-option, short the K_3-option.

Arbitrage exists if we can find $x_i > 0, \forall i \in \{1, 2, 3\}$ such that

$$x_1 C_1 + x_2 C_2 - x_3 C_3 = 0$$

and

$$f(S) = x_1 (S - K_1)^+ + x_2 (S - K_2)^+ - x_3 (S - K_3)^+ \geq 0, \forall S \geq 0$$

As before, we can get

$$x_1 + x_2 - x_3 \geq 0$$

and

$$x_3 = x_1 \frac{C_1}{C_3} + x_2 \frac{C_2}{C_3} > x_1 + x_2$$

which can not be simultaneously satisfied.

In conclusion, in order to rule out the arbitrage opportunities,

$$\frac{C_1 - C_3}{C_1 - C_2} > \frac{K_3 - K_1}{K_3 - K_2}$$

has to be true.

Method 2 (Convexity Approach): The arbitrage opportunity is based on the fact that the convexity of the call option price with respect to strike is violated.

A function $f : \mathbb{R} \mapsto \mathbb{R}$ is strictly convex(upward) if and only if

$$f(\lambda x + (1 - \lambda) y) < \lambda f(x) + (1 - \lambda) f(y), \forall x, y \in \mathbb{R}, \forall \lambda \in (0, 1)$$

Suppose that there exist two strikes $0 < K_1 < K_3$ and a parameter $\lambda \in (0, 1)$ such that

$$C(\lambda K_1 + (1 - \lambda) K_3) \geq \lambda C(K_1) + (1 - \lambda) C(K_3)$$

where $C(K)$ is the call price at time zero with strike K.

We can construct the following portfolio T:

- Long λ call options with strike K_1
- Long $1 - \lambda$ call options with strike K_3
- Short one call option with strike $K_2 = \lambda K_1 + (1 - \lambda) K_3$

The cost for such portfolio is $\lambda C(K_1) + (1-\lambda) C(K_3) - C(K_2) \leq 0$ and at maturity, the value of the portfolio $V(T)$ is given by

$$V(T) = \begin{cases} 0 & S(T) \leq K_1 \\ \lambda (S(T) - K_1) & K_1 < S(T) \leq K_2 \\ \lambda (S(T) - K_1) - (S(T) - K_2) & K_2 < S(T) \leq K_3 \\ \lambda (S(T) - K_1) - (S(T) - K_2) + (1-\lambda)(S(T) - K_3) & K_3 < S(T) \end{cases}$$

and can be simplified as

$$V(T) = \begin{cases} 0 & S(T) \leq K_1 \\ \lambda (S(T) - K_1) > 0 & K_1 < S(T) \leq K_2 \\ (1-\lambda)(K_3 - S(T)) \geq 0 & K_2 < S(T) \leq K_3 \\ 0 & K_3 < S(T) \end{cases}$$

which says the portfolio has non-negative payoff at options maturity regardless of the state of the market and has positive payoffs for certain market states.

In the $(K, C(K))$ space, the straight line that passes through the points (K_1, C_1) and (K_3, C_3) is

$$y = \frac{C_1 (K_3 - x)}{K_3 - K_1} + \frac{C_3 (x - K_1)}{K_3 - K_1}$$

The point corresponding to K_2 on this line is $\frac{C_1(K_3-K_2)}{K_3-K_1} + \frac{C_3(K_2-K_1)}{K_3-K_1}$ and then if

$$C_2 \geq \frac{C_1 (K_3 - K_2)}{K_3 - K_1} + \frac{C_3 (K_2 - K_1)}{K_3 - K_1}$$

an arbitrage opportunity would then be present. Thus

$$C_2 < \frac{C_1 (K_3 - K_2)}{K_3 - K_1} + \frac{C_3 (K_2 - K_1)}{K_3 - K_1}$$